Journal of Statistical Physics, Vol. 117, Nos. 1/2, October 2004 (© 2004)

Is the Two-Dimensional One-Component Plasma Exactly Solvable?

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Received January 28, 2004; accepted April 1, 2004

The model under consideration is the two-dimensional (2D) one-component plasma of point-like charged particles in a uniform neutralizing background, interacting through the logarithmic Coulomb interaction. Classical equilibrium statistical mechanics is studied by non-traditional means. The question of the potential integrability (exact solvability) of the plasma is investigated, first at arbitrary coupling constant Γ via an equivalent 2D Euclidean-field theory, and then at the specific values of $\Gamma = 2*$integer via an equivalent 1D fermionic model. The answer to the question in the title is that there is strong evidence for the model being not exactly solvable at arbitrary Γ but becoming exactly solvable at $\Gamma = 2*$integer. As a by-product of the developed formalism, the gauge invariance of the plasma is proven at the free-fermion point $\Gamma = 2$; the related mathematical peculiarity is the exact inversion of a class of infinite-dimensional matrices.

KEY WORDS: Coulomb systems; one-component plasma; logarithmic interaction; field representation; gauge invariance.

1. INTRODUCTION

In this paper, we consider a classical (i.e. non-quantum) model which belongs to the general class of two-dimensional (2D) Coulomb systems of charged particles. According to the laws of 2D electrostatics, the particles can be thought of as infinitely long charged lines which are perpendicular to the confining surface. Thus, the electrostatic potential v at a

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point $\mathbf{r}$, induced by a unit charge at the origin, is given by the 2D Poisson equation

$$\Delta v(\mathbf{r}) = -2\pi\delta(\mathbf{r}) \tag{1.1}$$

In a plane, the solution of this equation, subject to the boundary condition $\nabla v(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$, reads

$$v(\mathbf{r}) = -\ln\left(\frac{|\mathbf{r}|}{r_0}\right) \tag{1.2}$$

where the free length constant r_0, which fixes the zero point of the potential, will be set for simplicity to unity. In the Fourier space, the Coulomb potential (1.2) exhibits the characteristic small-k behavior $\hat{v}(\mathbf{k}) = 1/|\mathbf{k}|^2$. This maintains many generic properties (like screening and the related sum rules[1]) of "real" 3D Coulomb fluids with the interaction potential $v(\mathbf{r}) = 1/|\mathbf{r}|$, $\mathbf{r} \in R^3$. The pair interaction energy of particles with charges q and q', localized at the respective positions $\mathbf{r}$ and $\mathbf{r}'$, is

$$v(\mathbf{r}, q; \mathbf{r}', q') = qq'v(|\mathbf{r} - \mathbf{r}'|) \tag{1.3}$$

A given continuous Coulomb system is classified via the number M of different mobile (point-like) species $\alpha = 1, 2, \ldots, M$, with the corresponding charges q_α and particle densities n_α, embedded in a fixed uniform background of charge density ρ_b. The most studied versions are the one-component plasma (OCP), or jellium, and the symmetric two-component plasma (TCP), sometimes called the Coulomb gas. In the OCP there is only one mobile species, $M = 1$, with $q_1 = q$ and $n_1 = n$, and neutralizing background of charge density ρ_b. It is useful to introduce the background "number density" n_b such that $\rho_b = -qn_b$; the neutrality condition is then equivalent to $n = n_b$. The symmetric TCP corresponds to $M = 2$, namely $q_1 = q$ and $n_1 = n/2$, $q_2 = -q$ and $n_2 = n/2$ (n stands for the total particle density), with no background charge $\rho_b = 0$. Due to the logarithmic nature of the interaction, the equilibrium statistical mechanics of the underlying 2D Coulomb systems at the inverse temperature $\beta = 1/(k_B T)$ depends exclusively on the dimensionless coupling constant $\Gamma = \beta q^2$; the particle density n only scales appropriately the distance. Both the OCP and TCP are solvable in the high-temperature Debye-Hückel limit $\Gamma \to 0$ (in the bulk[2] as well as for finite systems[3,4]) and at the free-fermion point $\Gamma = 2$ (see reviews[5,6]). Through a simple scaling argument, the density derivatives of the Helmholtz free energy can all be calculated exactly at arbitrary Γ. For instance, the exact equation of state for the pressure P, $\beta P = n(1 - \Gamma/4)$, has been known for a long time[7]. On the other hand, the temperature derivatives of the free energy, such as the internal energy or the specific heat, are highly nontrivial quantities which were obtained only in the stability region $\Gamma < 2$ of the 2D TCP by exploring the equivalent sine-Gordon model (for a short review, see ref. [8]).

The 2D OCP, which will be of interest in this paper, is formally related to the eigenvalue distribution of certain complex random matrices[9] and to the modulus squared of Laughlin's wave functions in the fractional quantum Hall effect[10,11]. There are indications from numerical simulations that, around $\Gamma \sim 142$, the fluid system undergoes a phase transition to a 2D Wigner crystal.[12] The existence of this transition has been put in doubt in a more recent paper.[13] As was already mentioned, by mapping onto free fermions the model is exactly solvable at the coupling $\Gamma = 2$, in the bulk[14] as well as in some inhomogeneous situations[5,6]. The other exact information comes from the sum rules for truncated particle correlations valid at arbitrary Γ of the fluid regime. The usual zeroth- and second-moment conditions,[1] having analogues in any dimension, are supplemented by the fourth-moment (compressibility) sum rule,[15] available explicitly due to the knowledge of the exact equation of state, and the sixth-moment condition,[16] related to universal finite-size properties of the Coulomb system.[17–19] At couplings $\Gamma = 2\gamma$ (γ a positive integer), the partition function of the 2D OCP confined to some special domains can, for $\gamma = 1, 2$ and 3, be calculated exactly up to a relatively large finite number of particles N. For γ being an odd integer, the methods based on the expansion of even powers of the Vandermonde determinant into Schur functions [20,21] turn out to be especially efficient. For γ being an even integer, representations based on the permutation group[22,23] are useful. At arbitrary integer γ, the 2D OCP is mappable onto a discrete 1D fermionic field theory.[24] Within this fermionic representation, a symmetry of the model with respect to a complex transformation of particle coordinates has been shown to imply a functional relation for the two-body density. The functional relation is equivalent to an infinite sequence of sum rules relating the coefficients of the short-distance expansion of the two-body density. The generalization of the symmetry to multi-particle densities, possessing a specific invariant structure, was presented in ref. 25.

The mathematical formulation of the 2D OCP looks at first sight simpler than the one of the 2D TCP. The mentioned integrability of the Coulomb gas therefore evokes the potential possibility of the integrability of the 2D OCP, and this is the main subject of the present paper. The integrability of the 2D jellium is investigated first at arbitrary coupling Γ via an equivalent 2D Euclidean-field theory, and then at special values

$\Gamma = 2*$integer via the equivalent 1D fermionic model introduced in ref. 24. As a by-product of the developed formalism, the gauge invariance of the 2D OCP is proven at the free-fermion point $\Gamma = 2$.

The paper is organized as follows.

Section 2 is devoted to the 2D Euclidean-field representation of the 2D OCP. In Subsection 2.1., we sketch briefly the phenomenological Debye-Hückel calculation of the free energy in order to have a test formula for functional methods. The 2D OCP is mapped onto a 2D Euclidean-field theory in Subsection 2.2. In Subsection 2.3., the "classical" integrability of the Euclidean-field representation of the 2D OCP is investigated by using a scheme proposed by Ghoshal and Zamolodchikov.(26)

Section 3 is devoted to a further development of the discrete 1D fermionic representation of the 2D OCP at couplings $\Gamma = 2*$integer.(24) At these couplings, the partition function of the plasma is shown to admit a representation in terms of a linear set of equations.

Section 4 deals with gauge invariance of the bulk 2D OCP at coupling $\Gamma = 2$ which has been proven previously by more standard methods in ref. 27. The alternative proof of gauge invariance presented here is related to the exact inversion of a class of infinite-dimensional matrices, which is of mathematical interest.

A brief recapitulation is given in Section 5.

2. 2D FIELD REPRESENTATION

2.1. Debye-Hückel Calculation

In the mean-field approximation, the effective electric potential ψ at distance r of charge q, placed at the origin $\mathbf{0}$ and surrounded by mobile q-charges plus the neutralizing background, is given by the 2D Poisson equation

$$\Delta \psi(\mathbf{r}) = -2\pi q \left\{ \delta(\mathbf{r}) + n \left[\mathrm{e}^{-\beta q \psi(\mathbf{r})} - 1 \right] \right\} \tag{2.1}$$

The mean-field Boltzmann factor can be linearized for high temperatures. Eq. (2.1) then transforms to

$$\left(\Delta - \kappa^2 \right) \psi(\mathbf{r}) = -2\pi q \delta(\mathbf{r}) \tag{2.2}$$

where κ is the inverse Debye length defined by $\kappa^2 = 2\pi \Gamma n$. The solution of (2.2) reads $\psi(\mathbf{r}) = q K_0(\kappa r)$, where K_0 is a modified Bessel function. The excess (i.e., over ideal) free energy per particle, f_{ex}, is expressible in terms of the excess potential energy per particle

$$u_{\text{ex}}(\beta) = \frac{q}{2} \lim_{r \to 0} [\psi(r) + q \ln r] \tag{2.3}$$

as follows

$$\beta f_{\text{ex}} = \int_0^\beta d\beta' u_{\text{ex}}(\beta') \tag{2.4}$$

Using the short-distance expansion of K_0,[28]

$$K_0(x) = -\ln(x/2) - C + O(x^2 \ln x) \tag{2.5}$$

C is the Euler's constant, we arrive at the expression

$$\beta f_{\text{ex}} \sim -\frac{\Gamma}{4} \ln\left(\frac{\kappa^2}{4}\right) + \frac{\Gamma}{4}(1 - 2C) \tag{2.6}$$

valid in the small coupling limit $\Gamma \to 0$.[29] In what follows, this will be a test formula for functional methods.

An analogous procedure can be applied to the $\Gamma \to 0$ limit of the 2D TCP of $\pm q$ charged particles. The excess free energy per particle is again obtained in the form (2.6).

2.2. Field-Theoretical Representation

The 2D Coulomb potential (1.2) is singular at $r = 0$. This causes mathematical difficulties when representing interacting Coulomb systems as equivalent field theories. To avoid this problem, we will consider the Coulomb potential regularized smoothly at short distances:

$$v_{\text{reg}}(r) = -\ln r - K_0\left(\frac{r}{\epsilon}\right), \qquad \epsilon > 0 \tag{2.7}$$

In 3D, the analogous regularization has been used in ref. 30. Since the Bessel function $K_0(x)$ decays to zero exponentially as $x \to \infty$, $v_{\text{reg}}(r)$ has the large-r asymptotic of the pure Coulomb potential. On the other hand, using the short-distance expansion (2.5) in (2.7), the self-energy is finite

$$v_{\text{reg}}(0) = C - \ln 2 - \frac{1}{2} \ln \epsilon^2 \tag{2.8}$$

It is easy to verify that the regularized Coulomb potential satisfies the differential equation

$$\left(\Delta - \epsilon^2 \Delta^2\right) v_{\text{reg}}(\mathbf{r}) = -2\pi\delta(\mathbf{r}) \tag{2.9}$$

which is the counterpart of the 2D Poisson equation (1.1).

We are interested in the bulk thermodynamic properties of the OCP, defined in the infinite region $\Lambda = R^2$ with the volume $|\Lambda| \to \infty$. For N mobile particles at positions $\{\mathbf{r}_j\}_{j=1}^N$ in Λ, we introduce the microscopic density of particles $\hat{n}$ and of the total charge $\hat{\rho}$ as follows

$$\hat{n}(\mathbf{r}) = \sum_{j=1}^{N} \delta(\mathbf{r} - \mathbf{r}_j), \quad \hat{\rho}(\mathbf{r}) = q \sum_{j=1}^{N} \delta(\mathbf{r} - \mathbf{r}_j) - q n_b \tag{2.10}$$

Here, $-qn_b$ is the fixed (i.e., N-independent) charge density of the background. The total interaction energy of the particle-background system is expressible as

$$E_N(\{\mathbf{r}_j\}) = \frac{1}{2} \int_\Lambda \mathrm{d}^2 r \int_\Lambda \mathrm{d}^2 r' \hat{\rho}(\mathbf{r}) v_{\text{reg}}(|\mathbf{r} - \mathbf{r}'|) \hat{\rho}(\mathbf{r}') - \frac{1}{2} N q^2 v_{\text{reg}}(0) \tag{2.11}$$

We will work in the grand canonical ensemble with the fixed background[31,32] and position-dependent fugacity $z(\mathbf{r})$ of particles. The grand partition function Ξ at inverse temperature β is defined as the sum over all N-particle states

$$\Xi[z] = \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{j=1}^{N} \left[d^2 r_j z(\mathbf{r}_j) \right] \mathrm{e}^{-\beta E_N(\{\mathbf{r}_j\})} \tag{2.12}$$

The multi-particle densities can be obtained as the functional derivatives of the generator Ξ with respect to $z(\mathbf{r})$; after the functional derivatives are done, the homogeneous regime with the uniform fugacity $z(\mathbf{r}) = z$ is considered. At the one-particle level, the particle density is given by

$$\begin{aligned} n &= \langle \hat{n}(\mathbf{r}) \rangle \\ &= z(\mathbf{r}) \frac{1}{\Xi} \frac{\delta \Xi}{\delta z(\mathbf{r})} \bigg|_{\text{uniform}} \end{aligned} \tag{2.13}$$

At the two-particle level, one introduces the two-body density

$$\begin{aligned} n_2(\mathbf{r},\mathbf{r}') &= \langle \hat{n}(\mathbf{r})\hat{n}(\mathbf{r}')\rangle - n\delta(\mathbf{r}-\mathbf{r}') \\ &= z(\mathbf{r})z(\mathbf{r}')\frac{1}{\Xi}\frac{\delta^2\Xi}{\delta z(\mathbf{r})\delta z(\mathbf{r}')}\bigg|_{\text{uniform}} \end{aligned} \tag{2.14}$$

The grand partition function (2.12) can be expressed in terms of a 2D Euclidean-field theory. We start by the standard procedure (see, e.g., ref. 33) and substitute the representation (2.11) of E_N in the Boltzmann factor $\exp(-\beta E_N)$. The self-energy term renormalizes the fugacity, $z(\mathbf{r}) \to \bar{z}(\mathbf{r}) = z(\mathbf{r})\exp[\Gamma v_{\text{reg}}(0)/2]$. According to relation (2.9), $-(\Delta - \epsilon^2\Delta^2)/(2\pi)$ is the inverse operator of v_{reg}. The bilinear term in $\exp(-\beta E_N)$ can thus be linearized by applying the Hubbard-Stratonovich transformation

$$\begin{aligned} &\exp\left[-\frac{\beta}{2}\int d^2r\int d^2r'\hat{\rho}(\mathbf{r})v_{\text{reg}}(|\mathbf{r}-\mathbf{r}'|)\hat{\rho}(\mathbf{r}')\right] \\ &\quad = \frac{\int\mathcal{D}\phi\exp\left\{\int d^2r\left[\frac{1}{2}\phi(\Delta-\epsilon^2\Delta^2)\phi + \mathrm{i}\sqrt{2\pi\beta}\phi\hat{\rho}\right]\right\}}{\int\mathcal{D}\phi\exp\left[\int d^2r\frac{1}{2}\phi(\Delta-\epsilon^2\Delta^2)\phi\right]} \end{aligned} \tag{2.15}$$

Here, $\phi(\mathbf{r})$ is a real scalar field with all derivatives vanishing at infinity and $\int\mathcal{D}\phi$ denotes the functional integration over this field. The terms $\phi\Delta\phi$ and $\phi\Delta^2\phi$ can be turned into $-|\nabla\phi|^2$ and $(\Delta\phi)^2$, respectively, after performing integrations by parts with vanishing contributions from infinity. Inserting $\hat{\rho}$ from (2.10), particle coordinates in (2.12) become decoupled from each other and one can sum over N, with the result

$$\begin{aligned} \Xi = \int\frac{\mathcal{D}\phi}{D}\exp\Bigg\{-\int d^2r\Bigg[&\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\epsilon^2(\Delta\phi)^2 - \bar{z}(\mathbf{r})\mathrm{e}^{\mathrm{i}\sqrt{2\pi\Gamma}\phi} \\ &+\mathrm{i}\sqrt{2\pi\Gamma}n_b\phi\Bigg]\Bigg\} \end{aligned} \tag{2.16}$$

where

$$D = \int\mathcal{D}\phi\exp\left\{-\int d^2r\left[\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\epsilon^2(\Delta\phi)^2\right]\right\} \tag{2.17}$$

is the normalization constant. In the homogeneous regime $\bar{z}(\mathbf{r}) = \bar{z}$, the uniform shift in ϕ

$$\phi(\mathbf{r}) \to \phi(\mathbf{r}) + \frac{\ln(\bar{z}/n_b)}{\mathrm{i}\sqrt{2\pi\Gamma}} \tag{2.18}$$

factorizes out the z-dependence of Ξ,

$$\Xi = \exp\left[|\Lambda|\ln\left(\frac{\bar{z}}{n_b}\right)+|\Lambda|n_b\right]\int\frac{\mathcal{D}\phi}{D}\exp\left\{-\int d^2r\left[\frac{1}{2}|\nabla\phi|^2+\frac{1}{2}\epsilon^2(\Delta\phi)^2+n_b\left(-\mathrm{e}^{\mathrm{i}\sqrt{2\pi\Gamma}\phi}+1+\mathrm{i}\sqrt{2\pi\Gamma}\phi\right)\right]\right\} \tag{2.19}$$

The particle density n is yielded by the homogeneous analogue of Eq. (2.13) as follows

$$n = z\frac{\partial}{\partial z}\left(\frac{\ln\Xi}{|\Lambda|}\right) = n_b \tag{2.20}$$

This means that from the grand partition sum (2.12) only the term with the strict system neutrality survives, in the spirit of ref. 31. The density-fugacity relationship is trivial, namely the density does not depend on the fugacity. This enables us to pass to the canonical ensemble via the Legendre transformation

$$-\beta F(n) = \ln\Xi - N\ln z \tag{2.21}$$

where F is the Helmholtz free energy and $N=n|\Lambda|$. The excess free energy, related to F as follows $-\beta F_{\mathrm{ex}} = -\beta F + N\ln n - N$, then reads

$$-\beta F_{\mathrm{ex}}(n) = \frac{N\Gamma}{2}\left(C-\ln 2-\frac{1}{2}\ln\epsilon^2\right)+\ln L \tag{2.22}$$

where we have substituted the explicit form of the self-energy (2.8) and grouped the field part into the quantity L defined by

$$L = \frac{\int\mathcal{D}\phi\exp\left\{-\int d^2r\left[\frac{1}{2}|\nabla\phi|^2+\frac{1}{2}\epsilon^2(\Delta\phi)^2+n\left(-\mathrm{e}^{\mathrm{i}\sqrt{2\pi\Gamma}\phi}+1+\mathrm{i}\sqrt{2\pi\Gamma}\phi\right)\right]\right\}}{\int\mathcal{D}\phi\exp\left\{-\int d^2r\left[\frac{1}{2}|\nabla\phi|^2+\frac{1}{2}\epsilon^2(\Delta\phi)^2\right]\right\}} \tag{2.23}$$

For the pure Coulomb interaction ($\epsilon=0$), the field representation of the free energy similar to the one described by Eqs. (2.22) and (2.23) was established directly in the canonical format by Brilliantov.[34] The problem of the divergent self-energy was incorrectly ignored there, although this quantity enters into the final formulae. In what follows, we aim at deriving the small-Γ expansion of $\ln L$ in (2.22) in order to show that, in the limit $\epsilon \to 0$, the divergent self-energy term $\propto \ln \epsilon^2$ is canceled and the Debye-Hückel result (2.6) is reproduced correctly. For small Γ, we expand the exponential

$$e^{i\sqrt{2\pi\Gamma}\phi} \sim 1 + i\sqrt{2\pi\Gamma}\phi - \frac{1}{2}(2\pi\Gamma)\phi^2 \tag{2.24}$$

L then becomes equal to

$$L = \frac{\int \mathcal{D}\phi \exp\left\{-\int d^2r \left[\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\epsilon^2(\Delta\phi)^2 + \frac{1}{2}\kappa^2\phi^2\right]\right\}}{\int \mathcal{D}\phi \exp\left\{-\int d^2r \left[\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\epsilon^2(\Delta\phi)^2\right]\right\}} \tag{2.25}$$

The Gaussian functional integrals can be diagonalized in the Fourier **k**-space, with the result

$$\begin{aligned} L &= \prod_k \left(\frac{k^2 + \epsilon^2 k^4}{\kappa^2 + k^2 + \epsilon^2 k^4} \right)^{1/2} \\ &= \exp\left\{ \frac{1}{2}|\Lambda| \int \frac{d^2k}{(2\pi)^2} \ln\left(\frac{k^2 + \epsilon^2 k^4}{\kappa^2 + k^2 + \epsilon^2 k^4} \right) \right\} \end{aligned} \tag{2.26}$$

The integration over **k** can be carried out explicitly and one finds

$$\ln L = \frac{|\Lambda|}{8\pi} \left(t_+ \ln t_+ + t_- \ln t_- + \frac{1}{\epsilon^2} \ln \epsilon^2 \right) \tag{2.27}$$

where

$$t_\pm = \frac{1 \pm \sqrt{1 - 4\epsilon^2\kappa^2}}{2\epsilon^2} \tag{2.28}$$

In the $\epsilon \to 0$ limit,

$$t_+ = \frac{1}{\epsilon^2} - \kappa^2 + O(\epsilon^2), \quad t_- = \kappa^2 + O(\epsilon^2) \tag{2.29}$$

Consequently,

$$\ln L \sim \frac{N\Gamma}{4}\left(\ln \epsilon^2 + \ln \kappa^2 - 1\right) \quad \text{as } \epsilon \to 0 \tag{2.30}$$

Inserting this into (2.22), the singular $\ln \epsilon^2$ term disappears and one recovers the Debye-Hückel result (2.6).

The two-body density can be obtained from the field representation (2.16) of $\Xi[z]$ using formula (2.14). The uniform shift in the ϕ-field, relation (2.18), then leads to

$$\frac{n_2(\mathbf{r},\mathbf{r}')}{n(\mathbf{r})n(\mathbf{r}')} = \frac{\langle e^{i\sqrt{2\pi\Gamma}\phi(\mathbf{r})} e^{i\sqrt{2\pi\Gamma}\phi(\mathbf{r}')}\rangle}{\langle e^{i\sqrt{2\pi\Gamma}\phi(\mathbf{r})}\rangle\langle e^{i\sqrt{2\pi\Gamma}\phi(\mathbf{r}')}\rangle} \tag{2.31}$$

where the averages $\langle \cdots \rangle$ are taken with the field action

$$S[\phi] = \int d^2r \left[\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\epsilon^2(\Delta\phi)^2 + n\left(-e^{i\sqrt{2\pi\Gamma}\phi} + 1 + i\sqrt{2\pi\Gamma}\phi\right)\right] \tag{2.32}$$

Note that because the self-energy does not enter explicitly into (2.31), one can put $\epsilon = 0$ in the action (2.32) (the consequent singularities in the numerator and the denominator must be precisely canceled with one another), and consider

$$S[\phi] = \int d^2r \left[\frac{1}{2}|\nabla\phi|^2 + n\left(-e^{i\sqrt{2\pi\Gamma}\phi} + 1 + i\sqrt{2\pi\Gamma}\phi\right)\right] \tag{2.33}$$

In the Debye-Hückel $\Gamma \to 0$ limit, the expansion of the exponential according to Eq. (2.24) transforms the action (2.33) to

$$S_{\mathrm{DH}} = \int d^2r \left[\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\kappa^2\phi^2\right] \tag{2.34}$$

Since $\langle\phi\rangle = 0$ with this action, the Wick's theorem for Gaussian integrals implies

$$\langle e^{i\sqrt{2\pi\Gamma}\phi(\mathbf{r})}\rangle = \exp\left\{-\pi\Gamma\langle\phi^2(\mathbf{r})\rangle\right\} \tag{2.35}$$

$$\langle e^{i\sqrt{2\pi\Gamma}[\phi(\mathbf{r})+\phi(\mathbf{r}')]}\rangle = \exp\left\{-\pi\Gamma\langle[\phi(\mathbf{r})+\phi(\mathbf{r}')]^2\rangle\right\} \tag{2.36}$$

Consequently,

$$\frac{n_2(\mathbf{r},\mathbf{r}')}{n^2} = \exp\left\{-2\pi\Gamma\langle\phi(\mathbf{r})\phi(\mathbf{r}')\rangle\right\} \tag{2.37}$$

For the quadratic action (2.34), the correlator $\langle\phi(\mathbf{r})\phi(\mathbf{r}')\rangle$ is equal to the inverse matrix element of the operator $-\nabla^2+\kappa^2$,

$$\langle\phi(\mathbf{r})\phi(\mathbf{r}')\rangle = \frac{1}{2\pi}K_0(\kappa|\mathbf{r}-\mathbf{r}'|) \tag{2.38}$$

Thence

$$\frac{n_2(\mathbf{r},\mathbf{r}')}{n^2} = \mathrm{e}^{-\Gamma K_0(\kappa|\mathbf{r}-\mathbf{r}'|)} \sim 1-\Gamma K_0(\kappa|\mathbf{r}-\mathbf{r}'|) \tag{2.39}$$

in agreement with the standard Debye-Hückel calculation (see, e.g., refs. 16 and 29).

2.3. Classical Non-Integrability

The field action of the 2D OCP (2.33) belongs to a more general class of actions possessing the local form

$$S[\phi] = \int d^2r\left[\frac{1}{2}(\partial_x\phi)^2+\frac{1}{2}(\partial_y\phi)^2+4V(\phi)\right] \tag{2.40}$$

where the factor 4 in the potential term appears for notation convenience. The exact solvability of a theory depends on the particular form of the potential V, which is in our case

$$V(\phi) = \frac{n}{4}\left(-\mathrm{e}^{\mathrm{i}\sqrt{2\pi\Gamma}\phi}+1+\mathrm{i}\sqrt{2\pi\Gamma}\phi\right) \tag{2.41}$$

It is well known[35] that the 2D Euclidean field theory (2.40), defined in the space of points $\mathbf{r}=(x,y)$, is the imaginary-time $(y=\mathrm{i}t)$ continuation of the equivalent real-time 1+1 dimensional quantum field theory with the action

$$S[\phi] = \int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dx\left[\frac{1}{2}(\partial_t\phi)^2-\frac{1}{2}(\partial_x\phi)^2-4V(\phi)\right] \tag{2.42}$$

This action is dominated by fields satisfying the "classical" equation of motion $\delta S = 0$. In terms of light-cone coordinates defined by $\partial_\pm = (\partial_t \pm \partial_x)/2$, the equation of motion reads

$$\partial_+ \partial_- \phi = -V'(\phi) \tag{2.43}$$

In the case of our potential (2.41), rescaling appropriately the ϕ-field, this equation is nothing but the $1+1$ analogue of the usual mean-field Poisson-Boltzmann equation for the OCP. In general, the integrability of a field theory is associated with existence of an infinite sequence of conserved quantities (integrals of motion or "charges"). In what follows, we use a general scheme[26] to find out whether or not there exists an infinite sequence of conserved quantities for our field theory with the potential V given by (2.41), in the classical limit, i.e., when the field ϕ is governed by the equation of motion (2.43). The scheme represents a unique way of determinig integrability properties of the given field theory.

Existence of a conserved charge is associated with the appearance of a pair of "conjugate" local field densities (T, θ), with zero boundary conditions at $x \to \pm\infty$, such that

$$\partial_- T = \partial_+ \theta \tag{2.44}$$

In terms of x and t variables this is equivalent to $\partial_t (T - \theta) = \partial_x (T + \theta)$. The integration over x results in

$$\frac{\partial}{\partial t} \left[\int_{-\infty}^{\infty} dx (T - \theta) \right] = (T + \theta)\big|_{-\infty}^{\infty} = 0 \tag{2.45}$$

and thence the charge $\int dx (T - \theta)$ is conserved. We look for (T, θ) as polynomial functions in derivatives $\partial_+ \phi, \partial_+^2 \phi$, etc. The notation T_s (θ_s) will be used for T (θ) with just s ∂_+-derivatives. Because of the specific form of the equation of motion (2.43), only T_{s+1} and θ_{s-1} can create the conjugate couple, $\partial_- T_{s+1} = \partial_+ \theta_{s-1}$. The conserved charge is then

$$Q_s = \int dx (T_{s+1} - \theta_{s-1}) \quad s = 1, 2, \ldots \tag{2.46}$$

Note that the total ∂_+ derivatives can be dropped from T because the consequent difference $\partial_+ - \partial_- = \partial_x$ produces vanishing boundary contributions to the conserved charge. Similarly, let $\tilde{T}_{s-1}$ and $\tilde{\theta}_{s+1}$ be the conjugate polynomials of given orders in ∂_--derivatives such that $\partial_-\tilde{T}_{s-1} = \partial_+\tilde{\theta}_{s+1}$. Then, the charge

$$\tilde{Q}_s = \int dx(\tilde{\theta}_{s+1} - \tilde{T}_{s-1}) \quad s = 1, 2, \dots \tag{2.47}$$

is conserved. As before, the total ∂_- derivatives can be dropped from $\tilde{\theta}$.

At $s = 1$, writing $T_2 = (\partial_+\phi)^2$ one has

$$\partial_- T_2 = 2(\partial_+\phi)(\partial_+\partial_-\phi) = -2(\partial_+\phi)V'(\phi) = \partial_+[-2V(\phi)] \tag{2.48}$$

so that $\theta_0 = -2V$ and

$$Q_1 = \int dx\left[(\partial_+\phi)^2 + 2V\right] \tag{2.49}$$

Analogously, $\tilde{\theta}_2 = (\partial_-\phi)^2$, $\tilde{T}_0 = -2V$ and

$$\tilde{Q}_1 = \int dx\left[(\partial_-\phi)^2 + 2V\right] \tag{2.50}$$

$Q_1 + \tilde{Q}_1$ and $Q_1 - \tilde{Q}_1$ are energy and momentum, respectively, and these two quantities are always conserved for any potential V. There is no solution for conjugate polynomials producing conserved charges at s being an even integer. For $s = 3$, there exist conjugate polynomials

$$T_4 = \left(\frac{b}{2}\right)^2 (\partial_+\phi)^4 + (\partial_+^2\phi)^2 \tag{2.51}$$

$$\theta_2 = -(\partial_+\phi)^2 V''(\phi) \tag{2.52}$$

and the corresponding $(\tilde{\theta}_4, \tilde{T}_2)$, provided that the potential satisfies the differential equation $V''' = b^2 V'$. This equation is fulfilled either for the trivial free field theory $(b = 0)$ or for the potential

$$V(\phi) = A\mathrm{e}^{b\phi} + B\mathrm{e}^{-b\phi}, \quad b \neq 0 \tag{2.53}$$

When the constants A and B are nonzero and equal to one another, one recognizes the sinh-Gordon (b real) or sine-Gordon (b imaginary) models. At $s=5$, there exist conjugate polynomials and the corresponding conserved charges if the potential is either of the previous form (2.53) or of the form

$$V(\phi)=A\mathrm{e}^{b\phi}+B\mathrm{e}^{-(b/2)\phi}, \quad b\neq 0 \tag{2.54}$$

This integrable field theory is known as the Bullough-Dodd model [36] and for imaginary b it corresponds to the $1:2$ charge-asymmetric Coulomb gas.[37] The models with potentials (2.53) and (2.54) are the only two one-component-field members of the integrable affine Toda field theories, based on the Dynkin-diagram classification of simple Lie groups. Programming the whole scheme in the symbolic language *Reduce*, we were able to proceed up to the relatively high $s=15$ order. Except for the repeated appearance of the two potentials (2.53) and (2.54), we did not find any other solution for the potential leading to conserved charges. We do not anticipate a sudden appearance of an additional potential producing conserved charges for $s>15$. If it is so the 2D OCP, characterized by the field potential (2.41), is not classically integrable.

The conjectured classical non-integrability does not exclude the complete "quantum" (all realizations of the ϕ-field are considered) integrability of the model. At specific values of the coupling constant Γ, quantum fluctuations of the field around its classical saddle-point value can make the plasma integrable, as it is at the free-fermion point $\Gamma=2$.

3. LINEAR FERMIONIC REPRESENTATION

We now consider the 2D OCP, confined to a domain Λ, directly in the canonical ensemble. N point-like q-charges are embedded in a spatially homogeneous background of charge density $\rho_b=-qn_b$. If the system-neutrality condition is imposed it holds $n_b=N/|\Lambda|$. The background produces the one-body electric potential $v_b(\mathbf{r})=\rho_b\int_\Lambda d^2r'v(|\mathbf{r}-\mathbf{r}'|)$ which satisfies the Poisson equation

$$\Delta v_b(\mathbf{r})=2\pi q n_b, \quad \mathbf{r}\in\Lambda \tag{3.1}$$

Since, written in the complex $(z,\bar{z})$-coordinates,

$$\Delta=4\partial_z\partial_{\bar{z}} \tag{3.2}$$

for a circularly symmetric background one has

$$v_{\text{circ}} = \text{const} + \frac{\pi q n_b}{2} z\bar{z} \tag{3.3}$$

The deformation of the circular boundary $\partial\Lambda$ or the presence of some charge densities outside of Λ generates an additional gauge potential v_{gauge} such that $\Delta v_{\text{gauge}}(z,\bar{z}) = 0$ for $z \in \Lambda$. With regard to (3.2), the general solution of this equation reads

$$v_{\text{gauge}}(z,\bar{z}) = A_0 + \sum_{s=1}^{\infty} \left(A_s z^s + B_s \bar{z}^s \right) \tag{3.4}$$

Since Coulomb potentials are real, physical situations correspond to the choice $A_s = \bar{B}_s$ for all s. The particular case of a quadrupolar potential $v_{\text{gauge}} = A(z^2 + \bar{z}^2)$ results from the deformation of the disk into an ellipse.[11,38]

The potential energy of N particles at positions $\{z_i \in \Lambda\}$ plus the background is

$$E = E_0 + q \sum_i v_b(z_i,\bar{z}_i) - q^2 \sum_{i<j} \ln |z_i - z_j| \tag{3.5}$$

The background-background interaction constant E_0 does not influence the particle densities and so it can be omitted. The partition function at inverse temperature β reads

$$Z_N = \frac{1}{N!} \int_\Lambda \prod_{i=1}^{N} \left[d^2 z_i \, w(z_i,\bar{z}_i) \right] \prod_{i<j} |z_i - z_j|^\Gamma \tag{3.6}$$

where $\Gamma = \beta q^2$ and the one-body Boltzmann factor $w(z,\bar{z}) = \exp[-\beta q v_b(z,\bar{z})]$. The multi-particle densities can be obtained in the standard way [see relations (2.13) and (2.14)]:

$$n(z,\bar{z}) = w(z,\bar{z}) \frac{1}{Z_N} \frac{\delta Z_N}{\delta w(z,\bar{z})} \tag{3.7}$$

$$n_2(z_1,\bar{z}_1|z_2,\bar{z}_2) = w(z_1,\bar{z}_1) w(z_2,\bar{z}_2) \frac{1}{Z_N} \frac{\delta^2 Z_N}{\delta w(z_1,\bar{z}_1) \delta w(z_2,\bar{z}_2)} \tag{3.8}$$

etc. We will study the special case of the plasma with a "soft wall" [39] when particles are confined by the background itself. In particular, for a fixed value of the particle number N one makes the radius of the homogeneous circular background infinite, $|\Lambda| \to \infty$. The particles will gather in a circular region of area N/n_b. In the limit $N \to \infty$, the soft-wall edge of this region will have the structure different from the one of the usual hard-wall problem, but the particle densities deep in the interior ($|z|$ finite) will not be touched by the soft wall and will attain the bulk values. Also the free energy will be modified only by a surface term. Within the soft-wall $|\Lambda| \to \infty$ formulation of the problem, we will consider two cases:

(i) the circularly symmetric homogeneous background, Eq. (3.3), with

$$w(z, \bar{z}) = \frac{1}{\pi} \exp\left(-\frac{\Gamma}{2}\pi n_b z\bar{z}\right) \tag{3.9}$$

(ii) the general homogeneous background with the gauge component (3.4), given by

$$w(z, \bar{z}) = \exp\left[-\frac{\Gamma}{2}\pi n_b z\bar{z} - \sum_{s=1}^{\infty}(a_s z^s + b_s \bar{z}^s)\right] \tag{3.10}$$

where $a_s = \beta q A_s$ and $b_s = \beta q B_s$. As was already mentioned above, real physical situations require $a_s = \bar{b}_s$ for all s and the values of gauge parameters must be such that the multiple integral determining the partition function (3.6) does not diverge.

For the coupling $\Gamma = 2\gamma$ (γ an integer), it has been shown in ref. 24 that the partition function (3.6) can be expressed as the integral over two sets of Grassmann variables $\{\xi_i^{(\alpha)}, \psi_i^{(\alpha)}\}$ each with γ components ($\alpha = 1, \ldots, \gamma$), defined on a discrete chain of N sites $i = 0, 1, \ldots, N-1$ and satisfying the ordinary anticommuting algebra[40], as follows:

$$Z_N(\gamma) = \int \mathcal{D}\psi \mathcal{D}\xi \ \mathrm{e}^{S(\xi,\psi)} \tag{3.11}$$

$$S(\xi, \psi) = \sum_{i,j=0}^{\gamma(N-1)} \Xi_i w_{ij} \Psi_j \tag{3.12}$$

Here, $\mathcal{D}\psi\mathcal{D}\xi = \prod_{i=0}^{N-1} d\psi_i^{(\gamma)} \ldots d\psi_i^{(1)} d\xi_i^{(\gamma)} \ldots d\xi_i^{(1)}$ and S involves pair interactions of "composite" operators

$$\Xi_i = \sum_{\substack{i_1,\ldots,i_\gamma=0 \\ (i_1+\cdots+i_\gamma=i)}}^{N-1} \xi_{i_1}^{(1)} \ldots \xi_{i_\gamma}^{(\gamma)}, \qquad \Psi_j = \sum_{\substack{j_1,\ldots,j_\gamma=0 \\ (j_1+\cdots+j_\gamma=j)}}^{N-1} \psi_{j_1}^{(1)} \cdots \psi_{j_\gamma}^{(\gamma)} \tag{3.13}$$

i.e., the products of all γ anticommuting-field components, belonging to either ξ- or ψ-set, with the fixed sum of site indices. The interaction strength is given by

$$w_{ij} = \int_\Lambda d^2z \; w(z,\bar{z}) z^i \bar{z}^j; \qquad i,j = 0,1,\ldots,\gamma(N-1) \tag{3.14}$$

Using the notation $\langle\cdots\rangle = \int \mathcal{D}\psi\mathcal{D}\xi e^S \cdots / Z_N(\gamma)$ for an averaging over the anticommuting variables, the particle density (3.7) and the two-body density (3.8) are expressible in the fermionic format as follows

$$n(z,\bar{z}) = w(z,\bar{z}) \sum_{i,j=0}^{\gamma(N-1)} \langle \Xi_i \Psi_j \rangle z^i \bar{z}^j \tag{3.15}$$

$$\begin{aligned} n_2(z_1,\bar{z}_1|z_2,\bar{z}_2) &= w(z_1,\bar{z}_1) w(z_2,\bar{z}_2) \\ &\times \sum_{i_1,j_1,i_2,j_2=0}^{\gamma(N-1)} \langle \Xi_{i_1} \Psi_{j_1} \Xi_{i_2} \Psi_{j_2} \rangle z_1^{i_1} \bar{z}_1^{j_1} z_2^{i_2} \bar{z}_2^{j_2} \end{aligned} \tag{3.16}$$

respectively.

The exact solvability of the 2D OCP at $\Gamma = 2$ $(\gamma = 1)$ is due to the bilinear form of $S = \sum_{i,j=0}^{\gamma(N-1)} \xi_i w_{ij} \psi_j$. Thus,

$$Z_N(\gamma = 1) = \mathrm{Det}(w_{ij})|_{i,j=0}^{N-1} \tag{3.17}$$

The two-correlators determining the particle density (3.15) are equal to the inverse elements of the $N \times N$ $\mathbf{w}$-matrix (3.14),

$$\langle \xi_i \psi_j \rangle = w_{ji}^{-1} \tag{3.18}$$

The Wick's theorem applied to the four-correlators in (3.16) implies

$$\begin{aligned} \langle \xi_{i_1} \psi_{j_1} \xi_{i_2} \psi_{j_2} \rangle &= \langle \xi_{i_1} \psi_{j_1} \rangle \langle \xi_{i_2} \psi_{j_2} \rangle - \langle \xi_{i_1} \psi_{j_2} \rangle \langle \xi_{i_2} \psi_{j_1} \rangle \\ &= w_{j_1 i_1}^{-1} w_{j_2 i_2}^{-1} - w_{j_2 i_1}^{-1} w_{j_1 i_2}^{-1} \end{aligned} \tag{3.19}$$

In the case of a circularly symmetric plasma with $w(\mathbf{r}) = w(r)$ and $\Lambda = \{r \le R\}$ [like the one of interest, defined by Eq. (3.9) and $R \to \infty$], the interaction matrix $\mathbf{w}$ with elements (3.14) becomes diagonal:

$$w_{ij} = w_i \delta_{ij}, \quad w_i = 2\pi \int_0^R dr\, w(r) r^{2i+1} \tag{3.20}$$

The "diagonalized" form of the partition function

$$Z_N(\gamma) = \int \mathcal{D}\psi\mathcal{D}\xi \prod_{i=0}^{\gamma(N-1)} \exp(\Xi_i w_i \Psi_i) \tag{3.21}$$

implies that only correlators $\langle \Xi_{i_1}\Psi_{j_1}\Xi_{i_2}\Psi_{j_2}\cdots\rangle$ with $i_1+i_2+\cdots = j_1+j_2+\cdots$ will be nonzero. The dependence of $Z_N(\gamma)$ on the set of moments $\{w_i\}_{i=0}^{\gamma(N-1)}$ is the crucial problem whose solution would mean the complete exact solution (free energy, correlation functions) of the bulk 2D OCP at the given coupling $\Gamma = 2\gamma$. Let us write down explicitly few examples of this dependence. At $\gamma = 1$, we have the simple result

$$Z_N(1) = w_0 w_1 \cdots w_{N-1} \tag{3.22}$$

At $\gamma = 2$, using the anticommuting integral rules one finds from (3.21) for small particle numbers $N = 2, 3$ that

$$Z_2(2) = w_0 w_2 + 2w_1^2 \tag{3.23}$$

$$Z_3(2) = w_0 w_2 w_4 + 2w_0 w_3^2 + 2w_1^2 w_4 + 4w_1 w_2 w_3 + 6w_2^3 \tag{3.24}$$

etc. At $\gamma = 3$ one has

$$Z_2(3) = w_0 w_3 + 3^2 w_1 w_2 \tag{3.25}$$

$$\begin{aligned} Z_3(3) = {} & w_0 w_3 w_6 + 3^2 w_0 w_4 w_5 + 3^2 w_1 w_2 w_6 \\ & + 6^2 w_1 w_3 w_5 + 15^2 w_2 w_3 w_4 \end{aligned} \tag{3.26}$$

etc. There exists one model exactly solvable for every γ and N, namely the 2D OCP constrained to a circle. In that case $w(r) = \delta(r-1)/(2\pi)$ and, consequently, $w_i = 1$ for all $i = 0, 1, \ldots, \gamma(N-1)$. It was proved in various ways[41] that

$$Z_N(\gamma) = \frac{(\gamma N)!}{(\gamma!)^N N!} \qquad \text{when all } w_i = 1 \tag{3.27}$$

Relations (3.22)-(3.26) pass this test.

We now aim at analyzing the structure of a general summand in $Z_N(\gamma)$. It follows directly from the fermionic representation (3.21) that each term is composed of just N w's, $w_{i_1} w_{i_2} \cdots w_{i_N}$. The transformation $\xi_i^{(\alpha)} \to \lambda^i \xi_i^{(\alpha)}$ for all $\alpha = 0, 1, \ldots, \gamma$ indices and sites $i = 0, 1, \ldots, N-1$

implies $\Xi \to \lambda^i \Xi$. As a consequence, the subscripts of the general term $w_{i_1} w_{i_2} \cdots w_{i_N}$ must satisfy the relation $\sum i = \gamma(0 + 1 + \cdots + N - 1) = \gamma N(N-1)/2$. It is necessary to distinguish between γ being an odd or even integer.

For γ an odd integer, the composite Ξ and Ψ operators are products of an odd number of anticommuting variables and so they themselves satisfy the usual anticommutation rules

$$\{\Xi_i, \Xi_j\} = \{\Psi_i, \Psi_j\} = \{\Xi_i, \Psi_j\} = 0 \tag{3.28}$$

for all $i, j = 0, 1, \ldots, \gamma(N-1)$. In particular it holds $\Xi_i^2 = \Psi_i^2 = 0$. The expansion of each exponential in (3.21) is thus

$$\exp(\Xi_i w_i \Psi_i) = 1 + \Xi_i w_i \Psi_i \tag{3.29}$$

and the partition function is represented as follows

$$Z_N(\gamma) = \sum_{\substack{i_1 < i_2 < \ldots < i_N = 0 \\ [\sum i = \gamma N(N-1)/2]}} C^2_{i_1, i_2, \ldots, i_N} w_{i_1} w_{i_2} \cdots w_{i_N} \tag{3.30}$$

$$C_{i_1, i_2, \ldots, i_N} = \int \mathcal{D}\xi \;\; \Xi_{i_1} \Xi_{i_2} \cdots \Xi_{i_N} \tag{3.31}$$

We see that for odd γ a given w_i can occur in a summand at most once.

When γ is an even integer, the composite Ξ and Ψ operators are products of an even number of anticommuting variables and so they commute with each other

$$[\Xi_i, \Xi_j] = [\Psi_i, \Psi_j] = [\Xi_i, \Psi_j] = 0 \tag{3.32}$$

for all $i, j = 0, 1, \ldots, \gamma(N-1)$. The expansion of an exponential is now a bit more complicated,

$$e^{\Xi_i w_i \Psi_i} = \begin{cases} \sum_{j=0}^{[2i/\gamma]+1} \frac{1}{j!} (\Xi_i w_i \Psi_i)^j & \text{for } i = 0, \ldots, \frac{\gamma}{2}(N-1) \\ \sum_{j=0}^{2N-1-[2i/\gamma]} \frac{1}{j!} (\Xi_i w_i \Psi_i)^j & \text{for } i = \frac{\gamma}{2}(N-1), \ldots, \gamma(N-1) \end{cases} \tag{3.33}$$

Next terms vanish because they contain second or higher power of at least one anticommuting ξ or ψ variable. Let us introduce the indices

$\{\alpha_i\}_{i=0}^{\gamma(N-1)}$ with the following value ranges:

$$\alpha_i = \begin{cases} 0, 1, \ldots, \left[\frac{2i}{\gamma}\right]+1 & \text{for } i=0, \ldots, \frac{\gamma}{2}(N-1) \\ 0, 1, \ldots, 2N-1-\left[\frac{2i}{\gamma}\right] & \text{for } i=\frac{\gamma}{2}(N-1), \ldots, \gamma(N-1) \end{cases} \tag{3.34}$$

The partition function for γ an even integer is then expressible as

$$Z_N(\gamma) = \sum_{\substack{\alpha_0, \cdots, \alpha_{\gamma(N-1)} \\ \left[\sum \alpha_i = N, \sum i\alpha_i = \frac{\gamma}{2}N(N-1)\right]}} C^2_{\alpha_0,\alpha_1,\ldots,\alpha_{\gamma(N-1)}} \frac{w_0^{\alpha_0} w_1^{\alpha_1} \cdots w_{\gamma(N-1)}^{\alpha_{\gamma(N-1)}}}{\alpha_0!\alpha_1! \cdots \alpha_{\gamma(N-1)}!} \tag{3.35}$$

where

$$C_{\alpha_0,\alpha_1,\ldots,\alpha_{\gamma(N-1)}} = \int \mathcal{D}\xi \;\; \Xi_0^{\alpha_0} \Xi_1^{\alpha_1} \cdots \Xi_{\gamma(N-1)}^{\alpha_{\gamma(N-1)}} \tag{3.36}$$

The underlying fermionic representations, relations (3.30), (3.31) for γ odd and relations (3.34)-(3.36) for γ even, contain the unknown coefficients which can be formally written in both cases as

$$C_{i_1,\ldots,i_N} = \int \mathcal{D}\xi \;\; \Xi_{i_1} \Xi_{i_2} \cdots \Xi_{i_N}, \qquad \sum i = \frac{\gamma}{2}N(N-1) \tag{3.37}$$

The "basic sector" of indices is

$$\begin{array}{l} i_1 < i_2 < \ldots < i_N, \text{ for } \gamma \text{ odd} \\ i_1 \le i_2 \le \ldots \le i_N, \text{ for } \gamma \text{ even} \end{array} \tag{3.38}$$

The C-coefficient with an arbitrary sequence of indices can be expressed in terms of the basic one with indices ordered according to (3.38) through a successive exchange of nearest-neighbor couples of composite operators,

$$C_{i_1,\ldots,i_j,i_{j+1},\ldots,i_N} = (-1)^\gamma C_{i_1,\ldots,i_{j+1},i_j,\ldots,i_N} \tag{3.39}$$

It is evident that

$$C_{0,\gamma,\ldots,\gamma(N-1)} = 1 \tag{3.40}$$

One can evaluate the C-coefficients directly from their definition (3.37) by using the rules of integration over anticommuting variables.

However, there exists another simpler way how to determine these coefficients. Let $P_{\gamma-1}(k,k')$ be a general polynomial of the $(\gamma-1)$ order in the k,k' variables, and consider this polynomial in the following combination with the composite operators

$$\sum_{\substack{k,k'=0\\(k+k'=K)}}^{\gamma(N-1)} P_{\gamma-1}(k,k')\,\Xi_k\,\Xi_{k'};\qquad K=0,1,\ldots,2\gamma(N-1) \tag{3.41}$$

Representing the Ξ-operators in terms of the anticommuting ξ-variables [see relation (3.13)], Eq. (3.41) can be rewritten as

$$\sum_{\substack{k_1,\ldots,k_\gamma\\k'_1,\ldots,k'_\gamma}=0}^{N-1} P_{\gamma-1}(k_1+\cdots+k_\gamma,k'_1 +\cdots+k'_\gamma)\xi^{(1)}_{k_1}\cdots\xi^{(\gamma)}_{k_\gamma}\xi^{(1)}_{k'_1}\cdots\xi^{(\gamma)}_{k'_\gamma}\delta_{K,k_1+\cdots k_\gamma+k'_1+\cdots k'_\gamma} \tag{3.42}$$

Since $P_{\gamma-1}$ is the polynomial of the $(\gamma-1)$ order in its arguments, at least one couple of the k,k' factors with the same subscript, say the couple k_1 and k'_1 associated with $\xi^{(1)}$, does not occur in the given term of the expansion of $P_{\gamma-1}(k_1+\cdots k_\gamma,k'_1+\cdots k'_\gamma)$. Thus, since

$$\sum_{k_1,k'_1=0}^{N-1}\xi^{(1)}_{k_1}\xi^{(1)}_{k'_1}\delta_{K,k_1+\cdots k_\gamma+k'_1+\cdots k'_\gamma} = \sum_{k_1<k'_1}\left(\xi^{(1)}_{k_1}\xi^{(1)}_{k'_1}+\xi^{(1)}_{k'_1}\xi^{(1)}_{k_1}\right)\delta_{K,k_1+\cdots k_\gamma+k'_1+\cdots k'_\gamma}=0 \tag{3.43}$$

the polynomial combination with composite operators (3.41) vanish. Passing from $P_{\gamma-1}(k,k')$ to $Q_{\gamma-1}(k+k',k-k')$ one has

$$\sum_{\substack{k,k'=0\\(k+k'=K)}}^{\gamma(N-1)} (k-k')^n\,\Xi_k\,\Xi_{k'}=0\qquad \text{for } n=0,1,\ldots,\gamma-1 \tag{3.44}$$

We remind that the Ξ-operators anticommute for γ odd and commute for γ even. This is why for γ odd only odd powers of $(k-k')$ provide a nontrivial information and for γ even only even powers of $(k-k')$ are

informative. In view of the representation (3.37), we finally arrive at a homogeneous set of linear equations for the coefficients C:
if γ is odd,

$$\sum_{\substack{k,k'=0 \\ (k+k'=K)}}^{\gamma(N-1)} (k-k')^n C_{i_1,\dots,i_{N-2},k,k'} = 0; \qquad n = 1, 3, \dots, \gamma-2 \tag{3.45}$$

if γ is even,

$$\sum_{\substack{k,k'=0 \\ (k+k'=K)}}^{\gamma(N-1)} (k-k')^n C_{i_1,\dots,i_{N-2},k,k'} = 0; \qquad n = 0, 2, \dots, \gamma-2 \tag{3.46}$$

In both cases K runs over $0, 1, \dots, 2\gamma(N-1)$.

In the first nontrivial case of $\gamma = 2$, according to Eq. (3.39) the interchange of subscripts does not change the sign of the C-coefficients. The normalization is $C_{0,2,\dots,2(N-1)} = 1$. Formula (3.46) with $n = 0$ applies. For $N = 2$, the normalization $C_{0,2} = 1$ is supplemented by the only equation

$$0 = C_{0,2} + C_{1,1} + C_{2,0} \tag{3.47}$$

so that $C_{1,1} = -2$. Inserting the two coefficients into the representation (3.35) one recovers the exact relation (3.23). For $N = 3$, the normalization $C_{0,2,4} = 1$ is supplemented by the (overcomplete) set of five equations

$$\begin{aligned}
0 &= C_{0,2,4} + C_{0,3,3} + C_{0,4,2} \\
0 &= C_{1,1,4} + C_{1,2,3} + C_{1,3,2} + C_{1,4,1} \\
0 &= C_{2,0,4} + C_{2,1,3} + C_{2,2,2} + C_{2,3,1} + C_{2,4,0} \\
0 &= C_{3,0,3} + C_{3,1,2} + C_{3,2,1} + C_{3,3,0} \\
0 &= C_{4,0,2} + C_{4,1,1} + C_{4,2,0}
\end{aligned} \tag{3.48}$$

which implies $C_{0,3,3} = C_{1,1,4} = -2$, $C_{1,2,3} = 2$ and $C_{2,2,2} = -6$. Inserting these coefficients into (3.35) one recovers the relation (3.24). We suggest that the sets of linear equations, (3.45) for γ odd and (3.46) for γ even, constitute, together with the relation (3.39) and the normalization (3.40), complete (more precisely, overcomplete) sets to be solved for the C-coefficients. We have checked this conjecture for $\gamma = 2$ ($\Gamma = 4$) up to $N = 12$ particles and for $\gamma = 3$ ($\Gamma = 6$) up to $N = 9$ particles.

In comparison with the technique of anticommuting variables, the algorithms for solving a set of linear equations are much faster. The presented scheme is therefore very convenient for exact computer calculations of the plasma with a finite number of particles. From the methods dealing with the finite number N of particles,[20–23] the closest one[23] also provides the representations of the partition function (3.30) for γ odd and (3.35) for γ even. The C-coefficients are expressed there as multiple integrals over the unit circle, and consequently as multiple summations over all possible permutations of N indices. Such algorithm is inferior to the present approach.

At this stage, we were not able to find a simplification of the underlying sets of linear equations in the thermodynamic limit. In spite of this the fact that the crucial C-coefficients are determined by linear equations may indicate the exact solvability of the 2D OCP at $\Gamma = 2{*}\text{integer}$.

4. GAUGE INVARIANCE

In the case of classical statistical systems with short-ranged interactions among constituents, the thermodynamic limit of an intensive quantity does not depend in general on the shape of the system and on the conditions at the boundary given by the surrounding medium. This may be no longer true for macroscopic systems with long-ranged interactions. A typical example is the shape-dependence of the dielectric susceptibility tensor for Coulomb conductors[42] caused by the long-range decay of charge correlations along the boundary.[43,44] Gauge invariance of Coulomb fluids is another, in a certain sense opposite, phenomenon related to the long-range nature of particle interactions. Let us consider a macroscopic Coulomb conductor in an arbitrary shaped domain Λ, with perhaps some charge distribution on the boundary $\partial\Lambda$. The effect of the boundary is then reflected in the bulk through a long-ranged one-body potential whose Laplacian is zero inside the domain Λ. The assumption of gauge invariance states that this one-body "gauge" potential is perfectly screened by the Coulomb system at macroscopic distances from the boundary, and so it does not affect the averaged particle distributions in the bulk interior.

The above developed fermionic formalism will be now used to prove gauge invariance of macroscopic Coulomb fluids at the special value of the coupling $\Gamma = 2$. The original proof of gauge invariance (up to the gauge potential being the polynomial of degree 2 in spatial x and y components) at $\Gamma = 2$ was presented in ref. 27 for a class of possibly inhomogeneous backgrounds. The present proof is an alternative one, applicable also to non-physical situations with a complex gauge potential.

At $\Gamma=2$ and in the units of $\pi n_b=1$, the one-body Boltzmann factor of the circularly symmetric background (3.9) defined in an infinite space reads

$$w(z,\bar{z})=\frac{1}{\pi}\mathrm{e}^{-z\bar{z}}, \qquad \Lambda=R^2 \tag{4.1}$$

The $N\times N$ matrix (3.14) becomes diagonal, $w_{ij}=w_i\delta_{ij}$, with elements

$$w_i=2\int_0^\infty dr\mathrm{e}^{-r^2}r^{2i+1}=i!; \qquad i=0,1,\ldots,N-1 \tag{4.2}$$

Combining Eqs. (3.15) and (3.18), the latter written as $\langle\xi_i\psi_j\rangle=\delta_{ij}/w_i$, the particle density at distance r from the center is given by

$$\frac{n(r)}{n_b}=\mathrm{e}^{-r^2}\sum_{i=1}^{N-1}\frac{r^{2i}}{i!} \tag{4.3}$$

In the limit $N\to\infty$ and for finite r, the bulk particle density is constant, $n(r)=n_b$, as was expected.

Let us now consider the general case of a homogeneous background characterized by the one-body Boltzmann factor (3.10) with gauge degrees of freedom,

$$w(z,\bar{z})=\frac{1}{\pi}\exp\left[-z\bar{z}-\sum_{s=1}^{\infty}(a_s z^s+b_s\bar{z}^s)\right] \tag{4.4}$$

The interaction (3.14) is non-diagonal,

$$w_{ij}=\int\frac{d^2z}{\pi}z^i\bar{z}^j\exp\left[-z\bar{z}-\sum_s(a_s z^s+b_s\bar{z}^s)\right] \tag{4.5}$$

$i,j=0,1,\ldots,N-1$. The strength of gauge parameters $\{a_s,b_s\}$ must be such that integrals in (4.5) converge. For a real gauge potential with $a_s=\bar{b}_s=a\mathrm{e}^{\mathrm{i}\phi}$, writing $z=r\mathrm{e}^{\mathrm{i}\varphi}$ we have $a_s z^s+b_s\bar{z}^s=2ar^s\cos(\phi+s\varphi)$. If $s\geq 3$, for a fixed ϕ there always exist the φ-angles such that $a\cos(\phi+s\varphi)<0$, which implies divergent $r\to\infty$ contributions to the integral in (4.5). Thence all gauge coefficients $\{a_s,b_s\}$ with $s\geq 3$ must be equal to zero in physically acceptable situations. For $s=2$, the convergence is ensured provided that $2a<1$.

In view of Eqs. (3.15) and (3.18), the particle density at point $(z,\bar{z})$ is given by

$$\frac{n(z,\bar{z})}{n_b} = \pi w(z,\bar{z}) \sum_{i,j=0}^{N-1} w_{ji}^{-1} z^i \bar{z}^j \tag{4.6}$$

In the thermodynamic limit $N \to \infty$, the bulk particle density is expected to be constant, $n(z,\bar{z}) = n_b$. This is true iff

$$\sum_{i,j=0}^{\infty} w_{ji}^{-1} z^i \bar{z}^j = \exp\left[z\bar{z} + \sum_s (a_s z^s + b_s \bar{z}^s) \right] \tag{4.7}$$

This equation can be understood as the generating relation, with z and $\bar{z}$ assumed as independent variables, for the inverse elements of the infinite w-matrix with elements (4.5).

To prove that the quantities w_{ji}^{-1} determined by Eq. (4.7) form indeed the inverse matrix of the one with elements (4.5) we have to show that it holds

$$\sum_{k=0}^{\infty} w_{ik} w_{kj}^{-1} = \delta_{ij}, \qquad \sum_{k=0}^{\infty} w_{ik}^{-1} w_{kj} = \delta_{ij} \tag{4.8}$$

for all $i,j = 0,1,\ldots$. Let us consider the auxiliary function

$$K_i(t) = \sum_{j,k=0}^{\infty} w_{ik} w_{kj}^{-1} t^j \tag{4.9}$$

Presuming the validity of the generating Eq. (4.7), a series of straightforward transformations yields

$$\begin{aligned} K_i(t) &= \sum_{j,k=0}^{\infty} \int \frac{d^2 z}{\pi} z^i \bar{z}^k w(z,\bar{z}) w_{kj}^{-1} t^j \\ &= \int \frac{d^2 z}{\pi} z^i \exp\left[-z\bar{z} + t\bar{z} - \sum_s a_s (z^s - t^s) \right] \\ &= \int \frac{d^2 z}{\pi} (z+t)^i \exp\left\{ -z\bar{z} - \bar{t} z - \sum_s a_s \left[(z+t)^s - t^s \right] \right\} \end{aligned} \tag{4.10}$$

Since $(z+t)^s - t^s = z^s + stz^{s-1} + \cdots + st^{s-1}z$, only terms t^i and $-z\bar{z}$ survive from $(z+t)^i$ and the exponential, respectively. Consequently,

$$K_i(t) = t^i \qquad \text{for all } i = 0, 1, \ldots \tag{4.11}$$

With regard to the definition (4.9) of $K_i(t)$, this equation implies the first relation in Eq. (4.8). The second relation can be obtained in a similar way by showing that $L_i(\bar{t}) = \sum_{j,k=0} w_{ik}^{-1} w_{kj} \bar{t}^j$ is equal to $\bar{t}^i$ for all $i = 0, 1, \ldots$. The proof is accomplished.

One can readily show that the proof of gauge invariance for the one-body density automatically ensures gauge invariance for the two-body density (3.16) with correlators (3.19).

Strictly speaking, gauge invariance has meaning only for real gauge potentials; the short discussion after formula (4.5) tells us that only gauge potential being polynomial of degree 2 in z and $\bar{z}$ complex coordinates is allowed in physical situations. On the other hand, the proof of the matrix inversion (4.8)–(4.11) requires only the finite values of matrix elements (4.5), without putting any further restriction on the $\{a_s, b_s\}$ gauge parameters which may therefore be unphysical. Typical unphysical examples leading to finite values of matrix elements are $b_s = 0$ or $a_s = b_s = \mathrm{i}$. It is interesting from a mathematical point of view that there exists a large family of infinite matrices, with elements w_{ij} $(i, j = 0, 1, \ldots)$ defined by Eq. (4.5), which are explicitly invertible by using the closed-form generating formula (4.7). A detailed analysis of this mathematical peculiarity goes beyond the scope of the present paper.

5. CONCLUSION

In the canonical ensemble, the 2D jellium is equivalent to the 2D Euclidean-field theory with the action (2.33). Here, the divergent Coulomb self-energy does not renormalize the model's parameters like it is in the sine-Gordon representation of the 2D Coulomb gas. In contrast to the sine-Gordon model, the quantum analogue of the present Euclidean theory is conjectured to be not integrable on the classical level (only such realizations of the ϕ-field are considered which minimalize the action) due to the lack of an infinite sequence of integrals of motion. The classical non-integrability does not exclude the complete quantum (all realizations of the ϕ-field are considered) integrability at specific discrete values of Γ, like it is at the free-fermion $\Gamma = 2$ point. This free-fermion coupling belongs to a family of couplings $\Gamma = 2\gamma$ (γ integer) which admit a 1D fermionic representation of the partition function. The fermionic representations, relations (3.30) and (3.31) for γ odd and relations (3.34)-(3.36) for

γ even, contain the unknown C-coefficients. These coefficients are determined by the homogeneous sets of linear equations, (3.45) for γ odd and (3.46) for γ even, supplemented by the exchange formula (3.39) and the normalization (3.40). This feature is a sign of integrability. The present analysis might be a challenge for specialists in the Field Theory.

The proof of gauge invariance of the 2D OCP at $\Gamma = 2$ is related to the exact inversion of a class of infinite-dimensional matrices which elements are determined by non-Gaussian integrals (4.5). This is interesting from a mathematical point of view.

ACKNOWLEDGMENTS

Section 2 aroused from stimulating discussions with B. Jancovici whom I am also thank for careful reading of the manuscript and very useful comments. I am grateful to P. Kalinay for computer calculations. The support by a VEGA grant is acknowledged.

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